Thermal Flexure Measurement and Inverse Characterization for a Tri-layer Thin Plate

Hua Lu, Alireza Shirazi*, Ahmad Varvani-Farahani

(Department of Mechanical and Industrial Engineering, Ryerson University,
350 Victoria Street, Toronto, Ontario M5B 2K3, Canada)

Abstract: The paper presents a new study on a tri-layer thin plate. Shadow moiré implemented with an advanced phase unwrapping technique is employed to obtain actual flexural deformation of a real-life plate sample subjected to thermal loads. An analytical model is re-formulated to provide the plate with global closed-form solutions of the plate deflection as well as the interfacial stress and strain. With the measurements and the solutions available, an inverse iterative approach is developed to evaluate and maximize the correlation between the measured and the predicted thermal flexure, leading to ascertained materials’ constitutive and thermal behaviour. The inverse search algorithm starts with estimated ranges of material property parameters and progressively updates them to finally approach the respective true values. The established model solutions along with the optimized material properties matrix enable an accurate evaluation of the interfacial stresses/strains for the specific plate sample.

Key words: Tri-layer thin plate, inverse method, shadow moiré, thermal warpage, interfacial stresses

0 Introduction

Since Timoshenko pioneered the study of stresses in bi-material thermostats[1], thermal behaviour of multi-layer plates or beams made of dissimilar layer materials has been a classic topic attracting the interests of generations of academics and industry researchers. The recently renewed enthusiasm on the trilayer plates[2,3] is driven by the surge of the applications of chemically and metallurgically bonded multi-layer joints to the areas of microelectronics, MEMS and nano-materials or nano-structures. Among reported investigations, a class of models for a tri-layer beam with a thin adhesive mid-layer offers global solutions for the interfacial stress and strain as well as the plate deflection. The solutions are analytically reached by introducing the terms of longitudinal and transverse interfacial compliances[4,5] These models have been captured attention for their convenience in application. However, there are some non-trivial points worth addressing from practical view point. First, necessary experimental validations have been seldom despite the restriction exerted by some
assumptions and simplifications made in the model development. Secondly, it is known that many constituent materials in such real assemblies barely behave linear-elastically due to their dependencies on time, temperature, geometric scale, manufacturing process and service environment. As a result, the actual properties of many materials are expected to deviate substantially from bulk tested generic values obtainable from published literatures\textsuperscript{16–18}. These inconsistent and widely variable constants challenge the proper applications of the modeling albeit the general understanding of the mechanism and the trend of the plate deformation that the modeling can provide.

This paper presents an application of Hybrid Experimental Analytical Inverse Method (HEAIM) to cope with the situation. The method employs an improved phase-shifted shadow moiré to ensure the accuracy of the thermal warpage measurement on real tri-layer plates. A newly modified tri-layer beam model giving closed form solutions for the thermal flexure and interfacial stress is the key of the method, which is also briefly introduced. The modified model has led to some pre-requisite conditions imposed by the previous model development relaxed and the free shear stress boundary condition satisfied at the edges. The analytical solutions enable an inverse analysis of warpage change in a temperature increment measured from a real assembly. Among input parameters of the inverse algorithm are the warpage measurements, the plate geometric dimensions as well as some known layer material properties. Unknown/unclear material properties are determined via an iterative approach, while the algorithm checks and maximizes the correlation between the experimentally measured and the analytically predicted warpage. The true values of these uncertain layer properties are simultaneously identified upon reaching a pre-set correlation criterion. The HEAIM’ s output parameters include the unknown elastic constants (\(E\) and \(\nu\)), the glass transition temperature (\(T_g\)) and the coefficient of thermal expansion (\(\text{CTE}\)), etc. Based on the fully optimized material property matrix and the solutions, the characterization for the tri-layer plate is completed with limited computation to determine the response parameters.

1 Advanced shadow moiré and thermal flexure measurement of tri-layer plates

The test system for the thermal warpage measurement is shown in Figure 1. The computer automated shadow moiré technique used was implemented with some major improvements\textsuperscript{18}. Firstly, a Least Squares (LS) algorithm for phase unwrapping was introduced to handle phase unwrapping induced error. The random phase error is mostly attributed to flaws in originally recorded fringe patterns. The error in isolated local sites can propagate during phase unwrapping and develop into an area defect in the final phase solutions, which is often seen in applying the Path Following unwrapping algorithm. The weighted Multi-grid Least-Squares algorithm (MGLS) adopted in this study has been proved insensitive to such random flaws. Figure 2 compares the phase solutions obtained by processing the same moiré fringes of a ball-grid array module of 0.6 mm solder ball pitch using the above mentioned unwrapping algorithms. The merit of LS algorithm in avoiding the random fringe flaws induced errors is clearly shown. In addition, LS algorithm imposes no restriction on the input data array dimensions. Unwrapping by using the algorithms of FFT (Fast Fourier Transform) and DCT (Discrete Cosine Transform) require both dimensions \(M\) and \(N\) of an input data array \(M \times N\) to be powers of two. A new system calibration was also implemented to make the system more convenient in coping with frequent system re-set requirement due to versatility of the applications. The new calibration uses a real-time image subtraction scheme that displays an image generated by subtracting a phase-shifted fringe pattern from its un-shifted counterpart. The artificial image is statistically analyzed and the resulted gray level distribution histogram (pixel number vs. gray-value)
is displayed along with the live subtraction image. The phase shift increases with the increasing out-of-plane translation of the grating. Before the fringe phase is artificially shifted, the peak of the histogram stays near zero in the grey scale axis. With increased phase shift, the peak moves along the axis, initially away from and later on back towards the origin. In the meantime the full darkness of the screen is gradually lost but also recovered again. As the peak of the histogram comes back to near the zero grey scale, which indicates the completion of the $2\pi$ phase shift, the micrometer reading gives the moiré grating translation, $W$. The system sensitivity factor is then readily determined via $w = W/2\pi$.

Directed by such a scheme, the new calibration can be conducted on the surface of a test sample without the need of using a known curved surface or a standard gauge block. The measurement accuracy is affected by the random CCD grey level reading error. The latter in turn is dependent on a combination of the system and environment related factors. The measurement sensitivity of phase-shifted shadow moiré of 1 $\mu$m has been claimed in literatures [e.g., 17]. The measurement repeatability was evaluated on our system for practical purposes, and the standard deviation of the surface height measurement was less than 10 $\mu$m at room temperature and 20 $\mu$m under elevated temperatures.

![Image](image_url)

**Figure 1** (a) Schematic for a phase-shifted shadow moiré testing system. 
(b) Photo of the phase-shifted shadow moiré system

![Image](image_url)

**Figure 2** Typical unwrapping results. Comparison shows the Path-following algorithm caused random phase defect to propagate into a wider area while no such similar phenomenon appeared when MGLS unwrapping algorithm was used.
Shadow moiré measures the distance from a surface point to the plane of the glass grating. A moiré fringe pattern represents the contour map of such surface height variation. Since the pattern varies depending on how the surface is oriented under the grating, a virtual reference plane is needed. The reference plane in this application is generated by a least squares difference (LSD) fitting to raw measurement data, as illustrated in Figure 3. The plane is further translated along the out-of-plane direction to make the final reference goes through the highest surface point if the warpage is concave or the lowest point if the plate is convex. The maximum warpage \( u_{\text{max}} \) and the peak-to-valley (P-V) warpage \( \Delta w \) characterise the state of the surface deflection. P-V warpage, namely \( \Delta w = u_{\text{max}} - u_{\text{min}} \), is the difference between the highest and the lowest surface points. \( \Delta w \) is always positive whereas \( u_{\text{min}} \) may change its sign to signal the surface inflection change. The two indices become identical in situations where only a single peak (or valley) is located across the measured surface. They may differ if a peak and a valley with opposite signs co-exist. A curve plotting \( u_{\text{min}} \) (or \( \Delta w \)) vs. temperature is usually used to characterize a plate’s thermal flexure behaviour since it well illustrates the trend of the plate warpage response to certain temperature profile. Other indices characterising the thermal flexure include the net warpage change with respect to a temperature increment, the stress-free temperature (the temperature when the surface shows flat) and the residual warpage after a temperature cycle, etc.

![Figure 3 Least Squares fitting generated new reference plane for warpage data representation](image)

2 Analytical model and solutions for tri-layer structure under thermal load

The model for a stack joined tri-layer plate as schematically shown in Figure 4 is constructed assuming uniform plate temperature, spherical bending shape, small slope change and constant through-thickness curvature. The interfacial compliances for either plane-strain or plane-stress condition are introduced and obtained using Ribière solution for a long-and-narrow strip\(^{[18-20]}\). The solutions shown and used below are developed in this study. The detailed formulation can be found in \(^{[21, 22]}\). The plate deflection (warpage) is given by

\[
\begin{align*}
\varepsilon(x, y) &= \frac{1}{(D_x + D_y)} \left( \frac{b_1}{2} + \frac{b_2}{2} \right) \\
&+ C_1 (x^2) + C_2 (x) + C_3 (y^2) + C_4 (y) + C_5 (x) + C_6 (y) \\
&+ C_7 (x^3) + C_8 (x^2 y) + C_9 (x y^2) + C_{10} (y^3)
\end{align*}
\] (1)
where

\[
C_1 = - \frac{1}{(D_k + D_h)} \left( \frac{h_1}{2} + \frac{h_2}{2} \right) \left\{ \begin{array}{l}
\frac{G_{11}}{2} \left( b^l + q \right) \left( \sin(ql) + p \cos(ql) \right) - \frac{G_{22}}{2} \left( b^l + q \right) \left( \sin(ql) - p \cos(ql) \right) \\
\frac{G_{11}}{2} \left( b^l - q \right) \left( \sin(ql) + p \cos(ql) \right) - \frac{G_{22}}{2} \left( b^l - q \right) \left( \sin(ql) - p \cos(ql) \right)
\end{array} \right\}
\]

\[
C_2 = 0
\]

\[
C_3 = - \frac{1}{(D_k + D_h)} \left( \frac{h_1}{2} + \frac{h_2}{2} \right) \left( \frac{G_{11}}{p^l + q} \left( p^l - 3 p q^l \right) + \frac{G_{22}}{p^l + q} \left( q^l - 3 p q^l \right) \right)
\]

\(D_k, h_i, l\) in the equation are respectively the flexural rigidity, the thickness and the half length of the layers. The subscript 1, 2, 3 is the layer number. \(G_{11}, G_{22}, p \) and \(q \) are given below.

The interfacial shear stress given below satisfies the zero stress at the plate’s free edges:

\[
\tau(x) = \frac{C_{11}}{p + q} \sinh(p x) \cos(q x) + C_{22} \cosh(p x) \sin(q x)
\]

(2)

where

\[
p = \frac{1}{2} \sqrt{B} \quad q = \frac{1}{2} \sqrt{B}
\]

\[
C_{11} = \frac{\Delta u \Delta T \cosh(p l) \sinh(q l)}{\Sigma} \quad \text{and} \quad C_{22} = - \frac{\Delta u \Delta T \sinh(p l) \cos(q l)}{\Sigma}
\]

\(\Sigma = p \sqrt{B} \cos(q l) \sinh(p l) + q \sqrt{B} \cosh(p l) \sinh(q l)\).

The interfacial peel stress \(p(x)\) is given as:

\[
p(x) = C_{11} \cosh(\Phi(x)) \cos(\Pi x) + C_{11} \sinh(\Phi(x)) \sin(\Pi x)
\]

(3)

where

\[
\Phi = \frac{1}{2} \sqrt{2} \sqrt{A + E} \quad \Pi = \frac{1}{2} \sqrt{2} \sqrt{A - E}
\]

The remaining constants/parameters appeared in the above equations, including \(A, B, E, F\), are given in Appendix I of this paper.

3 Characterisation for a tri-layer plate and determination of material parameters by HEIAM

HEAIM\(^{20,21}\) performs an inverse analysis of plate warpage within a temperature increment \(\Delta T\). The same analysis repeats in every increment to cover an entire thermal process in which a nonlinear thermal flexure is normally seen for a polymeric and alloy made tri-layer plate. The thin rectangular shaped plate as shown in Figure 4 has four symmetric axes including \(X\) and \(Y\) axis and two diagonals. The plane strain condition prevails in the cross-sections along the coordinate axes and is nearly satisfied along the diagonals (\(\Psi\) axes). As the model applied is plane strain one, the following analysis is restricted to these cross-sections. Fig. 5 gives a flowchart for the application of HEAIM. In the end of each iteration, a correlation coefficient \(R^2\) is generated. \(R^2\) is defined as follows:

\[
R^2 = 1 - \frac{SSerr}{SStot}
\]

(4)

where the subscript index \(\ast\) denotes the \(\ast\)th iteration. The right-hand term is the ratio of the total variance of the experimental data from the predicted values (\(SSerr\)) to the total of variability of the experimental data to its average value (\(SStot\)), and

\[
SSerr = \sum_{i=1}^{n} \Delta^2 = (w_{\ast i}(x_i, y) - w_i(x_i, y))^2
\]

(5)
Figure 4  A 3D Schematic of the model of a warped trilayer plate with cross-sections of symmetry shown

Figure 5  Flowchart of Hybrid Experimental Analytical Inverse Method (HEAIM)
\[
S_{\text{Stat.}} = \sum_{i=1}^{n} \left[ u_{m}(x_i, y) - \frac{1}{n} \sum_{i=1}^{n} u_{\text{exp}}(x_i, y) \right]^2
\]

In the above equations \( u_{m}(x_i, y) \) and \( u_{\text{exp}}(x_i, y) \) are respectively the model predicted and the measured warpage at a node \((x_i, y)\). For the cross section along \( X \)-axis, \( y=l \), for the cross section along \( Y \), \( y=y_i \). \( n \) is the number of data points (nodes). The warpage data arrays used in the analysis are taken from the respective two-dimensional matrices of the raw moiré measurements.

The search for the maximum correlation begins with an estimated range for each of the unknown variable of the material parameters. Material parameters with highly defined true values such as those for silicon can be treated as constants. A search step is set to divide each such range into equal increments. The correlation calculation changes one parameter per loop of calculation. For \( n \) equal steps, there will be \( n \) round of calculations in a loop, generating \( n \) different correlation coefficients. At the completion of the \( n \)th round, the highest correlation coefficient (\( R \)) is identified and the material parameters array is updated with the updated value of this parameter. The search for optimizing the next parameter goes on in the next iteration and so on the iterations for all other parameters. The search stops when the updated correlation coefficient reaches the pre-set criterion (85\% in this study). For \( m \) different unknown parameters a maximum of \( n^m \) correlation calculations are performed. The algorithm will also decide to refine the search step or modify the search ranges if the search fails to meet the criterion in the end of \( n^m \) calculations.

4 An example of application to a trilayer sample under thermal process

The trilayer test specimens were designed to closely simulate a real module, which is composed of a silicon IC (Integrated Circuits) chip as the top layer, an adhesive/underfill material as the mid-layer and a composite laminate substrate as the bottom layer. These samples, as the pictures in Figure 6 show, feature thin thickness and high mismatches between layers’ thermal expansion and elastic stiffness. The ratio of the thickness to in-plane dimensions was made small to ensure the validity of the theory for slender beams. The silicon layer and the substrate (VT47 E-glass composite laminate) were the commercially available thin sheets. The adhesive layer was made of HYSOL P4531 liquid underfill material. During the plate bonding, the capillary effect was utilized to lead the liquid adhesive to penetrate and fill the pre-set gap between the top and the bottom layers. As the mid-layer was cured in an oven the tri-layer plate was finally formed. The curing process was tested and optimized to control the air bubbles capturing in the layer. Presented in Table 1 are some of the material properties collected before application of HEAIM and HEAIM determined values, including elastic modulus \( E \), Poisson’s ratio \( \nu \) and coefficient of thermal expansion \( a \) as well as glass transition temperature (\( \text{Tg} \)) of substrates and underfills and curing conditions of underfills.

![Figure 6](image-url)  (a) Photo showing a test sample being prepared; (b) Photo showing the test sample is cut to the final size
<table>
<thead>
<tr>
<th>Temperature</th>
<th>3D Contour Plots</th>
<th>Diagonal Line Scans, Corner to Corner</th>
</tr>
</thead>
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<tr>
<td>20°C</td>
<td><img src="image1" alt="3D Contour Plot" /></td>
<td><img src="image2" alt="Diagonal Line Scan" /></td>
</tr>
<tr>
<td></td>
<td>$\Delta w (\text{mm}) = 0.133$</td>
<td></td>
</tr>
<tr>
<td>55°C</td>
<td><img src="image3" alt="3D Contour Plot" /></td>
<td><img src="image4" alt="Diagonal Line Scan" /></td>
</tr>
<tr>
<td></td>
<td>$\Delta w (\text{mm}) = 0.108$</td>
<td></td>
</tr>
<tr>
<td>100°C</td>
<td><img src="image5" alt="3D Contour Plot" /></td>
<td><img src="image6" alt="Diagonal Line Scan" /></td>
</tr>
<tr>
<td></td>
<td>$\Delta w (\text{mm}) = 0.070$</td>
<td></td>
</tr>
<tr>
<td>Temperature</td>
<td>3D Contour Plots</td>
<td>Diagonal Line Scans, Corner to Corner</td>
</tr>
<tr>
<td>-------------</td>
<td>------------------</td>
<td>-------------------------------------</td>
</tr>
<tr>
<td>130°C</td>
<td><img src="image1" alt="3D Contour Plot" /></td>
<td><img src="image2" alt="Line Scans" /></td>
</tr>
<tr>
<td></td>
<td>$\Delta w (\text{mm}) = 0.048$</td>
<td>Sample 17</td>
</tr>
<tr>
<td>160°C</td>
<td><img src="image3" alt="3D Contour Plot" /></td>
<td><img src="image4" alt="Line Scans" /></td>
</tr>
<tr>
<td></td>
<td>$\Delta w (\text{mm}) = 0.030$</td>
<td>Sample 17</td>
</tr>
<tr>
<td>217°C</td>
<td><img src="image5" alt="3D Contour Plot" /></td>
<td><img src="image6" alt="Line Scans" /></td>
</tr>
<tr>
<td></td>
<td>$\Delta w (\text{mm}) = 0.020$</td>
<td>Sample 17</td>
</tr>
</tbody>
</table>
Figure 7 3-D plot of warped surface (left) and warpage variation along diagonals (right) at 240°C for sample #17

Table 1 Material properties collected before HEAIM and determined by HEAIM

<table>
<thead>
<tr>
<th>Materials Properties</th>
<th>Substrate Laminate VT-47</th>
<th>Underfill HYSOL FP4531</th>
<th>Silicon Die</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Before HEAIM</td>
<td>After HEAIM</td>
<td>Before HEAIM</td>
</tr>
<tr>
<td>Temp. (°C)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E tensile (GPa)</td>
<td>24.5</td>
<td>23~24</td>
<td>N/A</td>
</tr>
<tr>
<td>Post $T_s$</td>
<td>15~17</td>
<td></td>
<td></td>
</tr>
<tr>
<td>CTE (ppm/°C)</td>
<td>11</td>
<td>6~7</td>
<td>28</td>
</tr>
<tr>
<td>Post $T_s$</td>
<td>13</td>
<td>10.5~11.5</td>
<td>104</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>0.18~0.19</td>
<td>0.23</td>
<td>N/A</td>
</tr>
<tr>
<td>$T_s$ (°C)</td>
<td>175~185</td>
<td>160 (zero stress temperature)</td>
<td>161</td>
</tr>
</tbody>
</table>

The thermally induced plate absolute/apparent warpage (measured on the substrate) varying with the temperature for a typical plate is shown in Figure 7. The data are presented in the forms of 3D surface plots and the variations along the diagonal cross-sections. The warping direction experiences a change from convex to concave during the temperature rise, as noted from Figure 7. The turning temperature for the adhesive layer, which roughly matches the glass transition $T_s$ of the adhesive, is considered closest to the stress-free temperature the plate experiences during the thermal excursion. Specifically for the sample #17 the $T_s$ was found to be around 160°C.

The properties of the substrate and underfill were determined based on the diagonal warpage data. The initially estimated ranges of these parameters such as the elastic constants and CTE were obtained from published literatures or from open sources such as materials’ suppliers. These values are usually considered as bulk parameters. Figure 8 compares the measured warpage with the final determined warpage by HEAIM for Sample #17 at $\Delta T = 30°C$ (from 100°C rising to 130°C). The comparison proves a high correlation (a factor greater than 95%) between the two.
Figure 8  Typical results of HAEIM predicted relative warpage along a diagonal compared with experimental measurements for Sample #17, $\Delta T=30^\circ C$ (100$^\circ C$ to 130$^\circ C$)

Figure 9  HAEIM determined temperature dependent material properties of a) elastic modulus of underfill material, b) elastic modulus of substrate material, and c) substrate CTE
The bilinear temperature variation of the underfill and substrate materials is revealed by the noticeable slope changes in the respective curves in Figure 9. This property is attributable to the materials’ glass transition temperature $T_g$. Table 1 gives the specific data as determined from this study. The thermal-mechanical properties of the layer materials have dominant impact on the warpage behaviour and the process induced interfacial residual stresses for a tri-layer plate\textsuperscript{[25–134]}. As an example, the thermal warpage normally experiences a sign change as the temperature rises from room temperature. It is commonly known that the change of the direction of the plate deflection at a temperature signifies a stress-free state of the plate is reached at the temperature point. The approximate range of this temperature can be estimated from the $T_g$ of the underfill and substrate materials as the result of applying HEAIM. Due to insufficient data at multiple temperature points, no exact $Z$ shaped variations are shown. However, the turning points around the respective $T_g$ points are obvious.

Additional verification was carried out by applying Digital Speckle Correlation to a small area of 0.5mm×0.5mm located at the mid-point of a plate edge with its center at $x=1$ and $y=0$, as shown in Figure 10. The area covers the full thickness of the adhesive layer (layer-2). Figure 11 gives a typical pattern of strain $\varepsilon_y$ directly measured using DSC when the temperature rose from 25℃ to 125℃. The measurements reveal the highest peel strain in that cross section occurs in the middle of the adhesive layer. The average normal strain within the small area covering the adhesive layer was calculated and compared with the predicted peel strain. A good agreement between the measured and the predicted peel strain in a temperature range from room up to 125℃ was reached as Figure 12 shows. Figure 13 shows the typical peel and shear stress varying along half of the interface for a sample.

![Figure 10](image.png)  
*Figure 10 The small area in the plate side edge where the DSC strain measurement was applied point*

5 Discussion and conclusions

By offering certain verified parametric solutions, a good analytical model for a tri-layer plate serves as an important guidance to understanding the plate behaviour. A valid application of the model solutions to evaluating a real tri-layer plate is, however, far from simply a numerical calculation due
mainly to the uncertainties in the material properties. The task however can be fulfilled with some combined inverse and forward approaches as well as dedicated experimental measurement. The moiré technique is a best fit for validating the global solutions of the model. Theories predicting materials’ failure are mostly strength or energy based. For a multi-layer plate in particular, the inter-layer de-bonding failure is attributed to the interfacial stresses. The deflection is an equally important response parameter of a tri-layer component owing to its impact on the component functioning and its effect on the interconnection with other components in a multi-level assembly. Due to a number of assumptions applied in the model development, the model validation is certainly a necessary step in a structural design evaluation that focuses on the interfacial mechanical reliability. While to measure the interfacial thermal stress in such thin cross-sections is enormously challenging, the measurement of the thermal strain is obviously more realistic due to the availability of several applicable techniques. Strain measurement at high special resolution takes some subtle experimental procedures, is usually limited

Figure 11  A typical 2D contour pattern representing DSC measured thermally induced strain in the area specified in Figure 10

Figure 12  Comparison between measured and predicted peel strain in the area specified in Figure 10
to local sites and often destructive. On the other hand, the full-field shadow moiré evaluation of the thermal deflection of these thin and multi-layer plates has evolved and become a practical and convenient tool to provide quantified visualization of the overall state of the plate deformation throughout a predetermined thermal process.

The new analytical solutions satisfies the force balance along the interfaces by that the interfacial peel stress is self-equilibrating and the force boundary condition by that the shear stress vanishes at the plate edges intersecting the interfaces. The study found that the predicted magnitude of the shear stress is greater than that of the peel stress, and both stresses are higher under room temperature than in the elevated temperatures, indicating the existence of significant process induced residual stress.
The inversely determined material properties along with the verified model in this study serve a benchmark example for the application of HEAIM. It is noted that the outcomes from such an application of HEAIM to a specific multi-layered plate are applicable to a class of plates with similar designs subject to similar processes. In such cases, the evaluation of a modified design or process likely can be accomplished with some simple calculations.

6 Acknowledgements

The financial supports to this study by NSERC and OCE are acknowledged.

Miss Ming Zhou conducted shadow moiré and DSC measurements and part of the data reduction used in this study.

References

Appendix-I

\[ C_3 = \frac{\left( h_2 - \frac{b_2}{2D_h} \right)}{\eta} \frac{\Delta \sigma T \cosh(p_1 \sin(q_1) \cos(q_1))}{\sum} - \frac{\Delta \sigma T \sinh(p_1 \cos(q_1)) \cosh(p_1 \sin(q_1))}{\sum} \]

\[ C_4 = \frac{\left( h_2 - \frac{b_2}{2D_h} \right)}{\eta} \frac{\Delta \sigma T \cosh(p_1 \sin(q_1) \cos(q_1))}{\sum} - \frac{\Delta \sigma T \sinh(p_1 \cos(q_1)) \cosh(p_1 \sin(q_1))}{\sum} \]

where

\[ \text{Constant}_1 = \frac{\left( \eta - \gamma \right) \cosh(\Phi) \cos(\Pi)}{\eta} - \left( \Phi - \Phi_1 \right) \cosh(\Phi) \cos(\Pi) \]

\[ + 2\Phi \sinh(\Phi) \sin(\Pi) \]

\[ \text{Constant}_2 = \frac{\left( \eta - \gamma \right) \sinh(\Phi) \sin(\Pi)}{\eta} - \left( \Phi - \Phi_1 \right) \sinh(\Phi) \sin(\Pi) \]

\[ - 2\Phi \cosh(\Phi) \cos(\Pi) \]

\[ \text{Constant}_3 = \frac{\left( \eta - \gamma \right) \Phi \sinh(\Phi) \cos(\Pi)}{\eta} - \left( \Phi - \Phi_1 \right) \Phi \sinh(\Phi) \cos(\Pi) \]

\[ + 2\Phi \text{cosh}(\Phi) \sin(\Pi) \]

\[ + 2\Phi \text{tanh}(\Phi) \cosh(\Phi) \cos(\Pi) \]

\[ \text{Constant}_4 = \frac{\left( \eta - \gamma \right) \Phi \cosh(\Phi) \sin(\Pi)}{\eta} - \left( \Phi - \Phi_1 \right) \Phi \cosh(\Phi) \sin(\Pi) \]

\[ + 2\Phi \text{tanh}(\Phi) \cosh(\Phi) \cos(\Pi) \]

\[ E = \frac{\left( \eta - \gamma \right)}{1 + \frac{1}{D}} \]

\[ A = \frac{\eta}{1 + \frac{1}{D}} \]

\[ G = \frac{h_2 - \frac{b_2}{2D_h}}{1 + \frac{1}{D_h}} \]

\[ B = \frac{\mu \beta - \mu \beta + \mu \beta}{(\lambda + \lambda) + \left( \frac{1}{D_h} + \frac{1}{D} \right) \left( \frac{h_2}{2} + \frac{h_2}{2} \right)} \]

\[ F = \frac{\mu \beta - \mu \beta + \mu \beta}{(\lambda + \lambda) + \left( \frac{1}{D_h} + \frac{1}{D} \right) \left( \frac{h_2}{2} + \frac{h_2}{2} \right)} \]